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# Graded PI-exponents of simple Lie superalgebras

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**Abstract.** We study  $\mathbf{Z}_2$ -graded identities of simple Lie superalgebras over a field of characteristic zero. We prove the existence of the graded PI-exponent for such algebras.

## 1. Introduction

Let  $A$  be an algebra over a field  $F$  with  $\text{char } F = 0$ . A natural way of measuring the polynomial identities satisfied by  $A$  is by studying the asymptotic behaviour of its sequence of codimensions  $\{c_n(A)\}$ ,  $n = 1, 2, \dots$ . If  $A$  is a finite dimensional algebra then the sequence  $\{c_n(A)\}$  is exponentially bounded. In this case it is natural to ask the question about existence of the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)} \quad (1)$$

called the PI-exponent of  $A$ . Such question was first asked for associative algebras by Amitsur at the end of 1980's. A positive answer was given in [6]. Subsequently it was shown that the same problem has a positive solution for finite dimensional Lie algebras [14], for finite dimensional alternative and Jordan algebras [5] and for some other classes. Recently it was shown that in general the limit (1) does not exist even if  $\{c_n(A)\}$  is exponentially bounded [15]. The counterexample constructed in [15] is infinite dimensional whereas for finite dimensional algebras the problem of the existence of the PI-exponent is still open. Nevertheless, if  $\dim A < \infty$  and  $A$  is simple then the PI-exponent of  $A$  exists as it was proved in [8].

If in addition  $A$  has a group grading then graded identities, graded codimensions and graded PI-exponents can also be considered. In this paper we discuss

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graded codimensions behaviour for finite dimensional simple Lie superalgebras. Graded codimensions of finite dimensional Lie superalgebras were studied in a number of papers (see for example, [11] and [12]). In particular, in [11] an upper bound of graded codimension growth was found for one of the series of simple Lie superalgebras.

In the present paper we prove that the graded PI-exponent of any finite dimensional simple Lie superalgebra always exists. All details concerning numerical PI-theory can be found in [7].

## 2. Main constructions and definitions

Let  $L = L_0 \oplus L_1$  be a Lie superalgebra. Elements from the component  $L_0$  are called *even* and elements from  $L_1$  are called *odd*. Denote by  $\mathcal{L}(X, Y)$  a free Lie superalgebra with infinite sets of even generators  $X$  and odd generators  $Y$ . A polynomial  $f = f(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathcal{L}(X, Y)$  is said to be a *graded identity* of Lie superalgebra  $L = L_0 \oplus L_1$  if  $f(a_1, \dots, a_m, b_1, \dots, b_n) = 0$  whenever  $a_1, \dots, a_m \in L_0, b_1, \dots, b_n \in L_1$ .

Denote by  $Id^{gr}(L)$  the set of all graded identities of  $L$ . Then  $Id^{gr}(L)$  is an ideal of  $\mathcal{L}(X, Y)$ . Given non-negative integers  $0 \leq k \leq n$ , let  $P_{k, n-k}$  be the subspace of all multilinear polynomials  $f = f(x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in \mathcal{L}(X, Y)$  of degree  $k$  on even variables and of degree  $n-k$  on odd variables. Then  $P_{k, n-k} \cap Id^{gr}(L)$  is the subspace of all multilinear graded identities of  $L$  of total degree  $n$  depending on  $k$  even variables and  $n-k$  odd variables. Denote also by  $P_{k, n-k}(L)$  the quotient

$$P_{k, n-k}(L) = \frac{P_{k, n-k}}{P_{k, n-k} \cap Id^{gr}(L)}.$$

Then the *partial* graded  $(k, n-k)$ -codimension of  $L$  is

$$c_{k, n-k}(L) = \dim P_{k, n-k}(L)$$

and the *total* graded  $n$ th codimension of  $L$  is

$$c_n^{gr}(L) = \sum_{k=0}^n \binom{n}{k} c_{k, n-k}(L). \quad (2)$$

If the sequence  $\{c_n^{gr}(L)\}_{n \geq 1}$  is exponentially bounded then one can consider the related bounded sequence  $\sqrt[n]{c_n^{gr}(L)}$ . The latter sequence has the following lower and upper limits

$$\underline{\exp}^{gr}(L) = \liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}, \quad \overline{\exp}^{gr}(L) = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}$$

called the *lower* and *upper* PI-exponents of  $L$ , respectively. If the ordinary limit exists, it is called the (ordinary) *graded PI-exponent* of  $L$ ,

$$\exp^{gr}(L) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}.$$

Symmetric groups and their representations play an important role in the theory of codimensions. In particular, in the case of graded identities one can consider the  $S_k \times S_{n-k}$ -action on multilinear graded polynomials. Namely, the subspace  $P_{k,n-k} \subseteq \mathcal{L}(X, Y)$  has a natural structure of  $S_k \times S_{n-k}$ -module where  $S_k$  acts on even variables  $x_1, \dots, x_k$  while  $S_{n-k}$  acts on odd variables  $y_1, \dots, y_{n-k}$ . Clearly,  $P_{k,n-k} \cap Id^{gr}(L)$  is the submodule under this action and we get an induced  $S_k \times S_{n-k}$ -action on  $P_{k,n-k}(L)$ . The character  $\chi_{k,n-k}(L) = \chi(P_{k,n-k}(L))$  is called  $(k, n-k)$  *cocharacter* of  $L$ . Since  $\text{char } F = 0$ , this character can be decomposed into the sum of irreducible characters

$$\chi_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} \chi_{\lambda,\mu} \quad (3)$$

where  $\lambda$  and  $\mu$  are partitions of  $k$  and  $n-k$ , respectively. All details concerning representations of symmetric groups can be found in [9]. An application of  $S_n$ -representations in PI-theory can be found in [1, ?, ?].

Recall that an irreducible  $S_k \times S_{n-k}$ -module with the character  $\chi_{\lambda,\mu}$  is the tensor product of  $S_k$ -module with the character  $\chi_\lambda$  and  $S_{n-k}$ -module with the character  $\chi_\mu$ . In particular, the dimension  $\deg \chi_{\lambda,\mu}$  of this module is the product  $d_\lambda d_\mu$  where  $d_\lambda = \deg \chi_\lambda$ ,  $d_\mu = \deg \chi_\mu$ . Taking into account multiplicities  $m_{\lambda,\mu}$  in (3) we get the relation

$$c_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} d_\lambda d_\mu. \quad (4)$$

A number of irreducible components in the decomposition of  $\chi_{k,n-k}(L)$ , i.e. the sum

$$l_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu}$$

is called the  $(k, n-k)$ -*colength* of  $L$ . The *total* graded colength  $l_n^{gr}(L)$  is

$$l_n^{gr}(L) = \sum_{k=0}^n l_{k,n-k}(L).$$

Now let  $L$  be a finite dimensional Lie superalgebra,  $\dim L = d$ . Then

$$c_n^{gr}(L) \leq d^n \quad (5)$$

by the results of [2] (see also [4]). On the other hand, there exists a polynomial  $\varphi$  such that

$$l_n^{gr} \leq \varphi(n) \quad (6)$$

for all  $n=1, 2, \dots$  as it was mentioned in [11]. Note also that  $m_{\lambda, \mu} \neq 0$  in (3) only if  $\lambda \vdash k, \mu \vdash n-k$  are partitions with at most  $d$  components, that is  $\lambda = (\lambda_1, \dots, \lambda_p), \mu = (\mu_1, \dots, \mu_q)$  and  $p, q \leq d = \dim L$ .

Since all partitions under our consideration are of the height at most  $d$ , we will use the following agreement. If say,  $\lambda$  is a partition of  $k$  with  $p < d$  components then we will write  $\lambda = (\lambda_1, \dots, \lambda_d)$  anyway, assuming that  $\lambda_{p+1} = \dots = \lambda_d = 0$ .

For studying asymptotic behaviour of codimensions it is convenient to use the following function defined on partitions. Let  $\nu$  be a partition of  $m$ ,  $\nu = (\nu_1, \dots, \nu_d)$ . We introduce the following function of  $\nu$ :

$$\Phi(\nu) = \frac{1}{\left(\frac{\nu_1}{m}\right)^{\frac{\nu_1}{m}} \dots \left(\frac{\nu_d}{m}\right)^{\frac{\nu_d}{m}}}.$$

The values  $\Phi(\nu)^m$  and  $d_\nu = \deg \chi_\nu$  are very close in the following sense.

**Lemma 2.1.** [8, Lemma 1] *Let  $m \geq 100$ . Then*

$$\frac{\Phi(\nu)^m}{m^{d^2+d}} \leq d_\nu \leq m\Phi(\nu)^m.$$

□

Function  $\Phi$  has also the following useful property. Let  $\nu$  and  $\rho$  be two partitions of  $m$  with the corresponding Young diagrams  $D_\nu, D_\rho$ . We say that  $D_\rho$  is obtained from  $D_\nu$  by pushing down one box if there exist  $1 \leq i < j \leq d$  such that  $\rho_i = \nu_i - 1, \rho_j = \nu_j + 1$  and  $\rho_t = \nu_t$  for all remaining  $1 \leq t \leq d$ .

**Lemma 2.2.** (see [8, Lemma 3], [16, Lemma 2]) *Let  $D_\rho$  be obtained from  $D_\nu$  by pushing down one box. Then  $\Phi(\rho) \geq \Phi(\nu)$ .*

□

### 3. Existence of graded PI-exponents

Throughout this section let  $L = L_0 \oplus L_1$  be a finite dimensional simple Lie superalgebra,  $\dim L = d$ . Then by (5) its upper graded PI-exponent exists,

$$a = \overline{exp}^{gr}(L) = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}.$$

Note that the even component  $L_0$  of  $L$  is not solvable since  $L$  is simple (see [13, Chapter 3, §2, Proposition 2]).

We shall need the following fact.

*Remark 3.1.* Let  $G$  be a non-solvable finite dimensional Lie algebra over a field  $F$  of characteristic zero. Then the ordinary PI-exponent of  $G$  exists and is an integer not less than 2.

*Proof.* It is known that  $c_n(G)$  is either polynomially bounded or it grows exponentially not slower than  $2^n$  (see [10]). The first option is possible only if  $G$  is solvable. On the other hand  $\exp(G)$  always exists and is an integer [14] therefore we are done.  $\square$

By the previous remark  $P_{n,0}(L) \gtrsim 2^n$  asymptotically and then

$$a \geq 2. \quad (7)$$

The following lemma is the key technical step in the proof of our main result.

**Lemma 3.2.** *For any  $\varepsilon > 0$  and any  $\delta > 0$  there exists an increasing sequence of positive integers  $n_0, n_1, \dots$  such that*

- (i)  $\sqrt[n]{c_n^{gr}(L)} > (1-\delta)(a-\varepsilon)$  for all  $n = n_q$ ,  $q = 1, 2, \dots$ ,
- (ii)  $n_{q+1} - n_q \leq n_0 + d$ .

*Proof.* Fix  $\varepsilon, \delta > 0$ . Since  $a$  is an upper limit there exist infinitely many indices  $n_0$  such that

$$c_{n_0}^{gr}(L) > (a-\varepsilon)^{n_0}.$$

Fixing one of  $n_0$  we can find an integer  $0 \leq k_0 \leq n_0$  such that

$$\binom{n_0}{k_0} c_{k_0, n_0-k_0}(L) > \frac{1}{n_0+1} (a-\varepsilon)^{n_0} > \frac{1}{2n_0} (a-\varepsilon)^{n_0} \quad (8)$$

(see (2)). Relation (6) shows that

$$\sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda, \mu} \leq \varphi(n)$$

for any  $0 \leq k \leq n$  where  $m_{\lambda, \mu}$  are taken from (3). Then (4) implies the existence of partitions  $\lambda \vdash k_0, \mu \vdash n_0 - k_0$  such that

$$\binom{n_0}{k_0} d_\lambda d_\mu > \frac{1}{2n_0 \varphi(n_0)} (a-\varepsilon)^{n_0}. \quad (9)$$

The latter inequality means that there exists a multilinear polynomial

$$f = f(x_1, \dots, x_{k_0}, y_1, \dots, y_{n_0-k_0}) \in P_{k_0, n_0-k_0}$$

such that  $F[S_{k_0} \times S_{n_0-k_0}]f$  is an irreducible  $F[S_{k_0} \times S_{n_0-k_0}]$ -submodule  $P_{k_0, n_0-k_0}$  with the character  $\chi_{\lambda, \mu}$  and  $f \notin Id^{gr}(L)$ . In particular, there exist  $a_1, \dots, a_{k_0} \in L_0, b_1, \dots, b_{n_0-k_0} \in L_1$  such that

$$A = f(a_1, \dots, a_{k_0}, b_1, \dots, b_{n_0-k_0}) \neq 0$$

in  $L$ . First we will show how to find  $n_1, k_1$  which are approximately equal to  $2n_0, 2k_0$ , respectively, satisfying the same inequality as (8).

Since  $L$  is simple and  $A \neq 0$  the ideal generated by  $A$  coincides with  $L$ . Clearly, every simple Lie superalgebra is centerless. Hence one can find  $c_1, \dots, c_{d_1} \in L_0 \cup L_1$  such that

$$[A, c_1, \dots, c_{d_1}, A] \neq 0$$

and  $d_1 \leq d-1$ . Here we use the left-normed notation  $[[a, b], c] = [a, b, c]$  for nonassociative products. It follows that a polynomial

$$[f_1, z_1, \dots, z_{d_1}, f_2] = g_2 \in P_{2k_0+p, 2n_0-2k_0+r}, \quad p+r = d_1,$$

is also a non-identity of  $L$  where  $z_1, \dots, z_{d_1} \in X \cup Y$  are even or odd variables, whereas  $f_1$  and  $f_2$  are copies of  $f$  written on disjoint sets of indeterminates,

$$f_1 = f(x_1^1, \dots, x_{k_0}^1, y_1^1, \dots, y_{n_0-k_0}^1),$$

$$f_2 = f(x_1^2, \dots, x_{k_0}^2, y_1^2, \dots, y_{n_0-k_0}^2).$$

Consider the  $S_{2k_0} \times S_{2n_0-2k_0}$ -action on  $P_{2k_0+p, 2n_0-2k_0+r}$  where  $S_{2k_0}$  acts on  $x_1^1, \dots, x_{k_0}^1, x_1^2, \dots, x_{k_0}^2$  and  $S_{2n_0-2k_0}$  acts on  $y_1^1, \dots, y_{n_0-k_0}^1, y_1^2, \dots, y_{n_0-k_0}^2$ . Denote by  $M$  the  $F[S_{2k_0} \times S_{2n_0-2k_0}]$ -submodule generated by  $g_2$  and examine its character. It follows from Richardson-Littlewood rule that

$$\chi(M) = \sum_{\substack{\nu \vdash 2k_0 \\ \rho \vdash 2n_0-2k_0}} t_{\nu, \rho} \chi_{\nu, \rho}$$

where either  $\nu = 2\lambda = (2\lambda_1, \dots, 2\lambda_d)$  or  $\nu$  is obtained from  $2\lambda$  by pushing down one or more boxes of  $D_{2\lambda}$ . Similarly,  $\rho$  is either equal to  $2\mu$  or  $\rho$  is obtained from  $2\mu$  by pushing down one or more boxes of  $D_{2\mu}$ . Then by Lemma 2.2 we have

$$\Phi(\nu) \geq \Phi(2\lambda) = \Phi(\lambda), \quad \Phi(\rho) \geq \Phi(2\mu) = \Phi(\mu).$$

By Lemma 2.1 and (9) we have

$$\binom{n_0}{k_0} (\Phi(\lambda)\Phi(\mu))^{n_0} > \frac{1}{2n_0^3 \varphi(n_0)} (a-\varepsilon)^{n_0}. \quad (10)$$

Now we present the lower bound for binomial coefficients in terms of function  $\Phi$ . Clearly, the pair  $(k, n-k)$  is a two-component partition of  $n$  if  $k \geq n-k$ . Otherwise  $(n-k, k)$  is a partition of  $n$ . Since  $x^{-x}y^{-y} = y^{-y}x^{-x}$  for all  $x, y \geq 0, x+y=1$ , we will use the notation  $\Phi\left(\frac{k}{n}, \frac{n-k}{n}\right)$  in both cases  $k \geq n-k$  or  $n-k \geq k$ . Then it easily follows from the Stirling formula that

$$\frac{1}{n} \Phi\left(\frac{k}{n}, \frac{n-k}{n}\right)^n \leq \binom{n}{k} \leq n \Phi\left(\frac{k}{n}, \frac{n-k}{n}\right)^n,$$

hence

$$\binom{qk_0}{qn_0} > \frac{1}{qn_0} \Phi\left(\frac{qk_0}{qn_0}, \frac{qn_0-qk_0}{qn_0}\right)^{qn_0} = \frac{1}{qn_0} \Phi\left(\frac{k_0}{n_0}, \frac{n_0-k_0}{n_0}\right)^{qn_0} \quad (11)$$

for all integers  $q \geq 2$  and also

$$\left(\Phi\left(\frac{k_0}{n_0}, \frac{n_0-k_0}{n_0}\right) \Phi(\lambda) \Phi(\mu)\right)^{n_0} > \frac{1}{2n_0^4 \varphi(n_0)} (a-\varepsilon)^{n_0}. \quad (12)$$

by virtue of (10).

Recall that we have constructed earlier a multilinear polynomial  $g_2 = [f_1, z_1, \dots, z_{d_1}, f_2]$  which is not a graded identity of  $L$  and  $f_1, f_2$  are copies of  $f$ . Applying the same procedure we can construct a non-identity of the type

$$g_q = [g_{q-1}, w_1, \dots, w_{d_{q-1}}, f_q]$$

of total degree  $n_{q-1} = n_{q-2} + n_0 + w_1 + \dots + w_{d_{q-1}}$  where  $d_{q-1} \leq d$  and  $f_q$  is again a copy of  $f$  for all  $q \geq 2$ .

As in the case  $q=2$  the  $F[S_{qk_0} \times S_{qn_0-qk_0}]$ -submodule of  $P_{k, n-k}(L)$  (where  $n = n_{q-1} = qn_0 + p'$ ,  $k = k_{q-1} = qk_0 + p''$ ) contains an irreducible summand with the character  $\chi_{\nu, \rho}$  where  $\nu \vdash qk_0, \rho \vdash qn_0 - qk_0$ ,  $\Phi(\nu) \geq \Phi(\lambda), \Phi(\rho) \geq \Phi(\mu)$ . Moreover, for  $n = n_{q-1}$  we have

$$\begin{aligned} c_n^{gr}(L) &\geq \binom{qn_0}{qk_0} d_\nu d_\rho > \frac{1}{n^{2d^2+2d}} \binom{qn_0}{qk_0} (\Phi(\lambda) \Phi(\mu))^{qn_0} \\ &> \frac{1}{n^{2d^2+2d+1}} \left(\Phi\left(\frac{k_0}{n_0}, \frac{n_0-k_0}{n_0}\right) \Phi(\lambda) \Phi(\mu)\right)^{qn_0} \end{aligned}$$

by Lemma 2.1 and the inequality (11). Now it follows from (12) that

$$c_n^{gr}(L) > \frac{1}{n^{2d^2+2d+1}} \frac{1}{(2n_0^4 \varphi(n_0))^q} (a-\varepsilon)^{qn_0}.$$

Note that  $qn_0 \leq n \leq qn_0 + qd$ . Hence  $q/n \leq 1/n_0$  and

$$(a - \varepsilon)^{qn_0} \geq \frac{(a - \varepsilon)^n}{a^{qd}}$$

since  $a \geq 2$  (see (7)). Therefore

$$\sqrt[n]{c_n^{gr}(L)} > \frac{(a - \varepsilon)^n}{n^{\frac{2d^2 + 2d + 1}{n}} (2a^d n_0^4 \varphi(n_0))^{\frac{1}{n_0}}}$$

for all  $n = n_q, q = 1, 2, \dots$ . Finally note that the initial  $n_0$  can be taken to be arbitrarily large. Hence we can suppose that

$$n^{-\frac{2d^2 + 2d + 1}{n}} (2a^d n_0^4 \varphi(n_0))^{-\frac{1}{n_0}} > 1 - \delta$$

for all  $n \geq n_0$ . Hence the inequality

$$\sqrt[n]{c_n^{gr}(L)} > (1 - \delta)(a - \varepsilon)^n$$

holds for all  $n = n_q, q = 0, 1, \dots$ . The second inequality  $n_{q+1} - n_q \leq n_0 + d$  follows from the construction of the sequence  $n_0, n_1, \dots$ , and we have thus completed the proof.  $\square$

Now we are ready to prove the main result of the paper.

**Theorem 3.3.** *Let  $L$  be a finite dimensional simple Lie superalgebra over a field of characteristic zero. Then its graded PI-exponent*

$$\exp^{gr}(L) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}$$

*exists and is less than or equal to  $d = \dim L$ .*

*Proof.* First note that, given a multilinear polynomial  $h = h(x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in P_{k, n-k}$ , the linear span  $M$  of all its values in  $L$  is a  $L_0$ -module since

$$\begin{aligned} [h, z] &= \sum_i h(x_1, \dots, [x_i, z], \dots, x_k, y_1, \dots, y_{n-k}) \\ &+ \sum_j h(x_1, \dots, x_k, y_1, \dots, [y_j, z], \dots, y_{n-k}) \end{aligned}$$

for any  $z \in \mathcal{L}(X, Y)_0$ . Hence  $ML_1 \neq 0$  in  $L$  and  $0 \equiv [h, w]$  is not an identity of  $L$  for odd variable  $w$  as soon as  $h \notin Id^{gr}(L)$ . It follows that

$$c_{k, n-k+1}(L) \geq c_{k, n-k}(L)$$



and then

$$c_n^{gr}(L) \geq c_m^{gr}(L) \quad (13)$$

for  $n \geq m$ .

Fix arbitrary small  $\varepsilon, \delta > 0$ . By Lemma 3.2 there exists an increasing sequence  $n_q, q=1, 2, \dots$ , such that  $c_n^{gr}(L) > ((1-\delta)(a-\varepsilon))^n$  for all  $n=n_q, q=0, 1, \dots$ , and  $n_{q+1} - n_q \leq n_0 + d$ . Denote  $b = (1-\delta)(a-\varepsilon)$  and take an arbitrary  $n_q < n < n_{q+1}$ . Then  $c_n^{gr}(L) \geq c_{n_q}^{gr}(L)$  and  $n - n_q \leq n_0 + d$ . Referring to (7) we may assume that  $b > 1$ . Then  $b^{n_q} \geq b^n \cdot b^{-(n_0+d)}$  and

$$c_n^{gr}(L) \geq (b^{1-\frac{n_0+d}{n}})^n$$

for all  $n_q \leq n \leq n_{q+1}$  and all  $q=0, 1, \dots$ , that is for all sufficiently large  $n$ . The latter inequality means that

$$\liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)} \geq (1-\delta)b = (1-\delta)^2(a-\varepsilon).$$

Since  $\varepsilon, \delta$  were chosen to be arbitrary, we have thus completed the proof of the theorem.  $\square$

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